THE ELECTROMAGNETIC FIELD IN A CAVITY UNDERGOING COMPRESSION

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The idea of producing extra-strong magnetic fields by compressing (imploding) conducting shells was put forward by Terletskii in [1]. In the experiments of [2 and 3] based on this method, field intensities on the order of 10^7 G were achieved. The plane problem of magnetic field compression without allowance for displacement currents in a vacuum was solved by Bichenkov [4]. We shall consider the plane and axisymmetrical problem of compression of a homogeneous magnetic field for the case of an ideally conducting boundary moving in accordance with a prescribed law. The method of integral transformations is used in solving both problems.

1. **Formulation of the plane problem and its solution by the method of characteristics.** Let us consider the propagation of plane electromagnetic waves in an infinite vacuum slot bounded by plane conductive walls $x = \pm l(t)$ (Fig. 1).

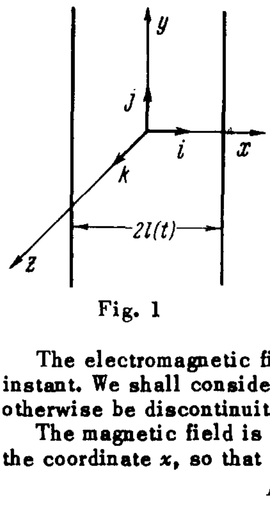

Fig. 1

Assuming that the electric and magnetic fields have the nonzero components $E = E_z\ (x, t)$, $H = H_y\ (x, t)$, we can write the wave propagation Eqs. as:

$$c\frac{\partial H}{\partial x} = \frac{\partial E}{\partial t}, \qquad c\frac{\partial E}{\partial x} = \frac{\partial H}{\partial t} \tag{1.1}$$

The electric field $\mathbf{E}' = \mathbf{E} + c^{-1}\,\mathbf{V}\times\mathbf{H}$ in the coordinate system connected with the moving wall must vanish at the wall.

For $x = \pm l(t)$ the velocity of the wall $\mathbf{V} = \pm l'(t)\,\mathbf{i}$. Hence,

$$E(\pm l(t),\ t) \pm \frac{l'(t)}{c} H(\pm l(t),\ t) = 0 \tag{1.2}$$

Thus, we must find the solution of system (1.1) in the domain $D\{0 < t < T,\ |x| < l(t)\}$ (where T is the time required for the walls to meet) which satisfies conditions (1.2) at the moving wall and the initial conditions $H(x,\ 0) = H_0$, $E(x,\ 0) = 0$.

The electromagnetic field outside the vacuum cavity is assumed to be zero at the initial instant. We shall consider $l(t)$ a decreasing function and assume that $l'(0) = 0$ (there will otherwise be discontinuities at the points $(\pm l(0),\ 0)$) and $|l'(t)| < c$.

The magnetic field is symmetrical and the electric field antisymmetrical with respect to the coordinate x, so that the solution of system (1.1) can be written as

$$E(x,\ t) = F(x + ct) - F(-x + ct)$$
$$H(x,\ t) = F(x + ct) + F(-x + ct)3) \tag{1.3}$$

Let $\xi = x + ct$, $\eta = x - ct$. From the initial conditions we find that

$$F(\xi) = {}^1/_2\ H_0,\ |\xi| < l_0$$

Thus, the solution of the problem is defined in domain 1 (Fig. 2) $\{|\xi| < l_0,\ |\eta| < l_0\}$. Since the solution in domain 1 is known, we can use boundary conditions (1.2) to find the solution in domain 2 $\{|\xi| < l_0,\ -x_1 < \eta < -l_0\}$ and in domain 3 $\{|\xi| < l_0,\ l_0 < \eta < x_1\}$, where $x_1 = l(t_1) + ct_1$, and t_1 is a root of Eq. $ct_1 = l(t_1) + l_0$.

Continuing this process, we can find the electromagnetic field distribution in the entire

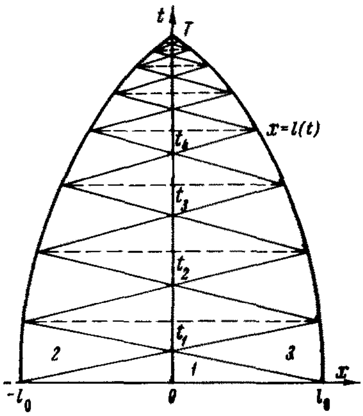

Fig. 2

domain D in a series of steps.

Let the function $p(\zeta)$ satisfy Eq.

$$l(p) + cp = \zeta \tag{1.4}$$

Let us determine x_k and t_k from the recurrent relations

$$x_k = l(t_k) + ct_k,$$

$$l(t_k) = -x_{k-1} + ct_k, \qquad t_0 = 0 \tag{1.5}$$

From boundary conditions (1.2) we find that

$$F(-l(t) + ct)(c - l'(t)) -$$
$$- F(l(t) + ct)(c + l'(t)) = 0 \tag{1.6}$$

Relation (1.6) enables us to determine the value of $F(\zeta)$ in the interval (x_k, x_{k+1}) if the values of this function are known in the interval (x_{k-1}, x_k). We set $F_k(\zeta) = F(\zeta)$ for $\zeta \in (x_{k-1}, x_k)$. We can show that

$$F_k(\zeta) = F_{k-1}(-\zeta + 2p(\zeta)) \frac{c - l'(p(\zeta))}{c + l'(p(\zeta))},$$

$$F_0(\zeta) = {}^1/_2 H_0 \tag{1.7}$$

The function $F(\zeta)$ defined by Formulas (1.7) is continuous in the interval $-l_0 < \zeta < cT$.

Let us denote the magnetic field at the wall by $H(l(t), t) = H^*(t)$. The function $F(\zeta)$ can be related to the field H^* at the boundary of the domain. In fact, from (1.3) and (1.7) we find that

$$F(\zeta) = {}^1/_2 H_0, \qquad -l_0 < \zeta < l_0$$

$$F(\zeta) = {}^1/_2 H^*(p(\zeta))\{1 - v(p(\zeta))\}, \qquad \zeta > l_0, \quad v(t) = c^{-1} l'(t) \tag{1.8}$$

2. Solution of the plane problem by the method of integral transformations. Let us apply the Fourier transform f with respect to the x coordinate to the functions E and H. This transform is given by

$$f\{E\} = g(k, t) = \int_{-l(t)}^{l(t)} E(x, t) e^{ikx} dx, \quad f\{H\} = h(k, t) = \int_{-l(t)}^{l(t)} H(x, t) e^{ikx} dx \tag{2.1}$$

$$E(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k, t) e^{-ikx} dk, \qquad H(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(k, t) e^{-ikx} dk \tag{2.2}$$

Application of the transform f to initial system (1.1) yields the following system for the Fourier representations g and h:

$$-ikg = c^{-1} \partial h / \partial t, \qquad -ikh = c^{-1} \partial g / \partial t + 2i \sin kl(t)(1 - v^2(t)) H^*(t)$$

Eliminating the electric field from the equations and boundary conditions, we obtain an Eq. for h,

$$\partial^2 h / \partial t^2 + c^2 k^2 h = \varphi(t, k), \qquad \varphi(t, k) = 2H^*(t) \sin kl(t)(1 - v^2(t)) c^2 k \tag{2.3}$$

The boundary conditions automatically enter into the equation. The solution of Eq. (2.3) is of the form

$$h(k, t) = h_1(k) e^{i\omega t} + h_2(k) e^{-i\omega t} + \int_0^t \frac{\varphi(\tau, k) \sin \omega(t - \tau)}{\omega} d\tau, \qquad \omega = ck \tag{2.4}$$

The functions $h_1(k)$ and $h_2(k)$ can be determined from the initial conditions

$$h_1(k) = h_2(k) = H_0 k^{-1} \sin kl_0 \tag{2.5}$$

The expression for $h(k, t)$ includes the unknown field $H^*(t)$ at the boundary. Its determination requires the use of inversion Formula (2.2), in which it is necessary to set $x = l(t)$. Here the integral in the right-hand side of the inversion formula must be doubled, since the field outside the vacuum cavity is assumed equal to zero and the Fourier integral converges at the point of discontinuity to its average value

$${}^1/_2\{H(l(t) - 0, t) + H(l(t) + 0, t)\} = {}^1/_2 H^*(t)$$

Let the function $\tau(t)$ be a root of Eq.

$$c\,(t-\tau)=l\,(t)+l\,(\tau) \tag{2.6}$$

$H^*(t)$ then satisfies the following functional Eq.:

$$H^*(t)=H_0\,[1+v\,(t)]^{-1}, \qquad 0<t<t_1$$
$$H^*(t)=H^*(\tau\,(t))\,\{1-v\,(\tau\,(t))\}[1+v\,(t)]^{-1}, \qquad t_1<t<T \tag{2.7}$$

The function $H^*(t)$ is known in the interval $(0,\ t_1)$. This implies that the function $H^*(\tau(t))$ is known in the interval $t_1<t<t_2$, since the values of τ run through the interval $(0,\ t_1)$, so that relation (2.7) makes it possible to determine $H^*(t)$ in the interval $(t_1,\ t_2)$. Continuing to determine the values of $H^*(t)$ in this way, we obtain recurrent formulas which enable us to determine the values of $H^*(t)$ in the interval $(t_k,\ t_{k+1})$ from the known values in the interval $(t_{k-1},\ t_k)$. The values of t_k are determined using Formulas (1.5). The process of solving functional Eq. (2.7) resembles the process of finding the function $F(\zeta)$ in solving the problem by the method of characteristics described in Section 1.

Let us define the sequence of functions $\tau_n(t)$ as follows:

$$\tau_n\,(t)=\tau_{n-1}\,\,(\tau\,(t)),\quad \tau_1\,(t)=\tau\,(t)$$

Then

$$H^*(t)=H^\circ(t)\quad (0<t<t_1),\qquad H^*(t)=H^\circ(t)\Pi_k\,(t)\quad (t_{k-1}<t<t_k)$$
$$H^\circ(t)=\frac{H_0}{1+v\,(t)}\,,\qquad \Pi_k=\prod_{n=1}^{k-1}W\,(\tau_n\,(t)),\qquad W\,(\tau)=\frac{1-v\,(\tau)}{1+v\,(\tau)} \tag{2.8}$$

We can prove the following statements regarding the qualitative features of the boundary function:

1°. The time sequence t_k defined by Formulas (1.5) tends to T as $k\to\infty$, while the sequence t_k-t_{k-1} tends to zero as $k\to\infty$.

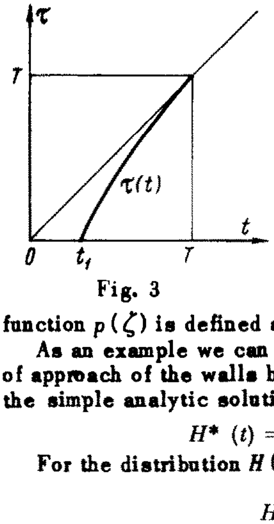

Fig. 3

2°. The function $\tau(t)$ defined in (2.6) differs from t by an arbitrarily small amount as $t\to T$ (Fig. 3).

3°. Let $l'(t)\leqslant 0$ and $l''(t)<0$ (in this case the time T required for the walls to meet is always finite). Then $H^*(t)$ increases monotonously with t, where $\lim H^*(t)=\infty$ as $t\to T$.

From the known boundary function $H^*(t)$ we can determine the magnetic field distribution $H(x,\ t)$ inside the cavity by making use of inversion Formula (2.2). We obtain

$$H\,(x,\ t)=F\,(x+ct)+F\,(-x+ct),$$
$$F\,(\zeta)={}^1\!/_2 H_0\quad (-l_0<\zeta<l_0)$$
$$F\,(\zeta)={}^1\!/_2 H^*\,(p\,(\zeta))\,\{1-v\,(p\,(\zeta))\}\quad (l_0<\zeta<cT)$$

The latter formula coincides exactly with (1.8), while the function $p(\zeta)$ is defined as the solution of Eq. (1.4).

As an example we can consider the case where beginning at the instant t_1 the velocity of approach of the walls becomes constant and equal to $v(t_1)=v$. In this case Eq. (2.7) has the simple analytic solution

$$H^*\,(t)=H^*\,(t_2)\,(T-t_2)\ /\ (T-t)\qquad (t_2<t<T)$$

For the distribution $H(x,\ t)$ we obtain Expression

$$H\,(x,\ t)=H^*\,(t_2)\,(1-v^2)\,\frac{(T-t_2)\,(T-t)}{(T-t)^2-x^2\,/\,c^2}$$
$$\{|\,x\,|\leqslant l\,(t),\ t_2<t<T\} \tag{2.9}$$

The ratio of the maximum value $H(x,\ t)=H^*(t)$ of the field with respect to x to the minimum value $H(0,\ t)$ with respect to x at the axis is $1/(1-v^2)$. For values of v^2 small compared to unity, the field inside the vacuum slot can be considered spatially homogeneous.

3. The axisymmetrical problem. Let the vacuum cavity be bounded by an infinite circular ideally conductive cylinder whose boundary moves in accordance with the prescribed law $R(t)$. Here we assume that $R(t)$ is a decreasing function and that $R'(0)=0$ and $|R'(t)|<c$. Assuming that the magnetic field has the nonzero axial component $H=H_z(r,\ t)$ and that the electric field has the axial component $E=-E_0(r,\ t)$, we arrive at the system

$$c\partial H / \partial r = \partial E / \partial t, \quad c\partial (rE) / \partial r = r\partial H / \partial t$$

with the condition

$$E(R(t), t) + R'(t)\, c^{-1} H(R(t), t) = 0$$

at the moving boundary and the initial conditions $H(r, 0) = H_0$ and $E(r, 0) = 0$.

By eliminating the electric field from the equations and boundary conditions, we can formulate the problem for the magnetic field alone. The problem then reduces to the solution of Eq.

$$\frac{\partial^2 H}{\partial t^2} = \frac{c^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right) \qquad \{0 < r < R(t),\ 0 < t < T\} \tag{3.1}$$

with the condition

$$c^2\left(\frac{\partial H}{\partial r}\right)_{r=R(t)} = -\frac{d}{dt}\left[R'(t) H(R(t), t)\right] - \frac{R'^2(t) H(R(t), t)}{R(t)} - R'(t)\left(\frac{\partial H}{\partial t}\right)_{r=R(t)}$$

at the moving boundary, and the initial conditions

$$H(r, 0) = H_0, \quad (\partial H / \partial t)_{t=0} = 0$$

To solve the problem we apply a finite Hankel transform of the form

$$h(\rho, t) = \int_0^{R(t)} H(r, t) J_0(\rho r) r\, dr, \qquad H(r, t) = \int_0^{\infty} h(\rho, t) J_0(\rho r) \rho\, d\rho \tag{3.2}$$

Here the field outside the cavity is assumed to be zero. The representation h of the field satisfies Eq.

$$\partial^2 h / \partial t^2 + c^2\rho^2 h = \varphi(\rho, t), \quad \varphi(\rho, t) = c^2 H^*(t)\rho R(t) J_1(\rho R(t)) \times$$

$$\times (1 - v^2(t)), \quad v(t) = c^{-1} R'(t), \quad H^*(t) = H(R(t), t) \tag{3.3}$$

With allowance for the boundary conditions, the solution of Eq. (3.3) is of the form

$$h(\rho, t) = \frac{H_0 R_0 \cos ct\rho J_1(\rho R_0)}{\rho} + \int_0^t \frac{\varphi(\rho, \tau)\sin c(t-\tau)\rho}{c\rho}\, d\tau \tag{3.4}$$

The function $\varphi(\rho, t)$ includes the unknown field $H^*(t)$ at the boundary of the domain. To determine this field we proceed as in the plane case, making use of the inversion formula from (3.2) in which we set $r = R(t)$. Doubling the integral in the right-hand side, we have

$$H^*(t) = 2H_0 R_0 \int_0^{\infty} J_0(\rho R(t)) J_1(\rho R_0) \cos ct\rho\, d\rho +$$

$$+ 2c \int_0^{\infty} \left\{\int_0^t H^*(\tau) R(\tau) J_1(\rho R(\tau)) [1 - v^2(\tau)] \sin c(t-\tau)\rho\, d\tau\right\} J_0(\rho R(t)) \rho\, d\rho \tag{3.5}$$

We can reverse the order of integration in the latter integral if by the double integral we mean the limit

$$\int_0^{\infty}\left(\int_0^t H^*(\tau) R(\tau) J_1(\rho R(\tau)) [1 - v^2(\tau)] \sin c(t-\tau)\rho\, d\tau\right) J_0(\rho R(t)) \rho\, d\rho =$$

$$= \lim_{b\to+0}\int_0^{\infty} e^{-b\rho}\left\{\int_0^t H^*(\tau) R(\tau) J_1(\rho R(\tau)) [1 - v^2(\tau)] \sin c(t-\tau)\rho\, d\tau\right\} J_0(\rho R(t)) \rho\, d\rho$$

We write

$$M(\rho, \tau) = H^*(\tau) R(\tau) [1 - v^2(\tau)] \sin c(t-\tau)\rho e^{-b\rho} J_0(\rho R(t)) J_1(\rho R(\tau))\rho.$$

Then, if $H^*(\tau)$ is integrable over the interval $(0, t)$, the integrals

$$\int_0^t M(\rho, \tau)\, d\tau, \qquad \int_0^{\infty} M(\rho, \tau)\, d\rho$$

converge uniformly – the first in ρ, $\rho \in (0, \infty)$, and the second in τ – over any compact interval which lies inside the interval $(0, t)$ and does not contain singularities of $H^*(t)$. For example, let $H^*(t)$ have a single singularity at $t = t^*$.

We stipulate that $\Delta(\delta)$ is a compact set $\{\Delta(\delta) = [0, t^* - \delta] + [t^* + \delta, t]\}$,

$$I(\delta, \rho) = \int\limits_{\Delta(\delta)} M(\rho, \tau)\, d\tau$$

Then

$$\int\limits_0^\infty I(\delta, \rho)\, d\rho$$

converges uniformly in δ, $0 < \delta < \delta_0$, where δ_0 is an arbitrarily small positive number. By virtue of the uniform convergence of the above integrals, we can assert that [5]

$$\int\limits_0^\infty \Big(\int\limits_0^t M(\rho, \tau)\, d\tau \Big) d\rho = \int\limits_0^t \Big(\int\limits_0^\infty M(\rho, \tau)\, d\rho \Big) d\tau \tag{3.6}$$

Making use of (3.5) and taking the limit as $b \to +0$ in (3.6), we obtain the following Volterra integral Eq. of the second kind for $H^*(t)$:

$$H^*(t) = B(t) + 2c \int\limits_0^t R(\tau) [1 - v^2(\tau)]\, S(t, \tau)\, H^*(\tau)\, d\tau \tag{3.7}$$

$$B(t) = 2H_0 R_0 \int\limits_0^\infty J_0(\rho R(t))\, J_1(\rho R_0) \cos ct\rho\, d\rho$$

$$S(t, \tau) = \lim_{b \to +0} \int\limits_0^\infty e^{-b\rho} \sin c(t - \tau)\rho J_0(\rho R(t))\, J_1(\rho R(\tau))\, \rho\, d\rho \tag{3.8}$$

In the classical sense the integral

$$\int\limits_0^\infty x J_0(\alpha x)\, J_1(\beta x) \sin \gamma x\, dx$$

diverges for all nonzero values of α, β, and γ. However, the above integral can be regularized by assigning to it some generalized value [6]

$$\Phi(\alpha, \beta, \gamma) = \lim_{b \to +0} \int\limits_0^\infty e^{-bx} J_0(\alpha x)\, J_1(\beta x)\, x \sin \gamma x\, dx$$

Considering the convergent integral

$$\Psi(\alpha, \beta, \gamma) = \int\limits_0^\infty J_0(\alpha x)\, J_0(\beta x) \sin \gamma x\, dx$$

we can prove that $\Phi = -\partial\Psi/\partial\beta$. The integral $\Psi(\alpha, \beta, \gamma)$ can be expressed in terms of Legendre functions of the first and second kind [7],

$$\Psi(\alpha, \beta, \gamma) = 0 \quad (0 < \gamma < \beta - \alpha,\ \beta > \alpha)$$

$$\Psi(\alpha, \beta, \gamma) = \frac{1}{2\sqrt{\alpha\beta}} P_{-1/2}(A) \quad (\beta - \alpha < \gamma < \beta + \alpha)$$

$$\Psi(\alpha, \beta, \gamma) = -\frac{1}{\pi\sqrt{\alpha\beta}} Q_{-1/2}(-A) \quad (\beta + \alpha < \gamma < \infty), \qquad A = (\beta^2 + \alpha^2 - \gamma^2)/2\alpha\beta$$

In this case the Legendre functions reduce to total elliptic integrals of the first kind, and we can show that

$$\Psi(\alpha, \beta, \gamma) = 0 \quad (0 < \gamma < \beta - \alpha), \qquad \Psi(\alpha, \beta, \gamma) = \frac{1}{\pi\sqrt{\alpha\beta}} K(k) \quad (\beta - \alpha < \gamma < \beta + \alpha)$$

$$\Psi(\alpha, \beta, \gamma) = \frac{1}{\pi\sqrt{\alpha\beta}} \mu K(\mu) \quad (\beta + \alpha < \gamma < \infty), \quad k = \Big(1 - \frac{(\alpha + \beta)^2 - \gamma^2}{4\alpha\beta}\Big)^{1/2}, \quad \mu = \frac{1}{k}$$

Hence, for $\Phi(\alpha, \beta, \gamma) = -\partial\Psi/\partial\beta$ we have Formula

$$\Phi(\alpha, \beta, \gamma) = 0 \qquad (0 < \gamma < \beta - \alpha)$$

$$\Phi(\alpha, \beta, \gamma) = \frac{1}{2\pi\beta\sqrt{\alpha\beta}}\left[K(k)\left(1 - \frac{\gamma^2+\beta^2-\alpha^2}{4\alpha\beta k^2}\right) + E(k)\frac{\gamma^2+\beta^2-\alpha^2}{4\alpha\beta k^2(1-k^2)}\right]$$

$$(\beta - \alpha < \gamma < \beta + \alpha)$$

$$\Phi(\alpha, \beta, \gamma) = \frac{\mu}{2\pi\beta\sqrt{\alpha\beta}}\left[K(\mu) - E(\mu)\frac{\gamma^2+\beta^2-\alpha^2}{4\alpha\beta(1-\mu^2)}\right] \qquad (\beta + \alpha < \gamma < \infty)$$

Let us now consider the integral

$$\Omega(\alpha, \beta, \gamma) = \int_0^\infty J_0(\alpha x) J_1(\beta x)\cos\gamma x\, dx$$

We can show that

$$\Omega(\alpha, \beta, \gamma) = \Omega(\alpha, \beta, 0) - \int_0^\gamma \Phi(\alpha, \beta, \nu)\, d\nu \tag{3.9}$$

Expression (3.9) will be used only for the values $\gamma < \alpha + \beta$. For $\gamma > \alpha + \beta$ the integral $\Omega(\alpha, \beta, \gamma)$ is computed in [8]. We obtain

$$\Omega(\alpha, \beta, \gamma) = 1/\beta \qquad (0 < \gamma < \beta - \alpha)$$

$$\Omega(\alpha, \beta, \gamma) = \frac{1}{\beta} + \frac{2}{\pi}\sqrt{\frac{\alpha}{\beta}}\int_0^k\left[K(\varkappa)\left(\frac{\nu^2+\beta^2-\alpha^2}{4\alpha\beta\varkappa^2} - 1\right) - E(\varkappa)\frac{\nu^2+\beta^2-\alpha^2}{4\alpha\beta\varkappa^2(1-\varkappa^2)}\right]\frac{\varkappa}{\nu}\, d\varkappa$$

$$\nu = \sqrt{(\beta-\alpha)^2 + 4\alpha\beta\varkappa^2} \qquad (\beta - \alpha < \gamma < \beta + \alpha)$$

$$\Omega(\alpha, \beta, \gamma) = -\frac{\beta}{2\gamma^2}F_4\left(\frac{3}{2}, 1; 1, 2; \frac{\alpha^2}{\gamma^2}, \frac{\beta^2}{\gamma^2}\right) \qquad (\beta + \alpha < \gamma < \infty)$$

Here $F_4(a_1, a_2; \delta, \delta'; x, y)$ is a generalized hypergeometric function of two variables [9].

The functions $B(t)$ and $S(t, \tau)$ defined in (3.8) can be expressed in terms of $\Phi(\alpha, \beta, \gamma)$ and $\Omega(\alpha, \beta, \gamma)$ as follows:

$$B(t) = 2H_0R_0\Omega(R(t), R_0, ct), \qquad S(t, \tau) = \Phi(R(t), R(\tau), c(t-\tau))$$

For $B(t)$ and $S(t, \tau)$ we obtain Formulas

$$B(t) = H_0\left(1 + \frac{4}{\pi}\sqrt{R_0R(t)}\int_0^{k^*}\left[K(\varkappa)\left(\frac{\vartheta^*}{\varkappa^2} - 1\right) - E(\varkappa)\frac{\vartheta^*}{\varkappa^2(1-\varkappa^2)}\right]\frac{\varkappa}{\nu^*}\, d\varkappa\right)$$

$$k^* = \left(1 - \frac{[R_0 + R(t)]^2 - c^2t^2}{4R_0R(t)}\right)^{1/2} \qquad (0 < t < t_1)$$

$$\vartheta^* = \frac{\nu^{*2} + R_0^2 - R(t)^2}{4R_0R(t)}, \qquad \nu^* = \sqrt{[R_0 - R(t)]^2 + 4\varkappa^2R_0R(t)}$$

$$B(t) = -H_0\frac{R_0^2}{c^2t^2}F_4\left(\frac{3}{2}, 1; 1, 2; \frac{R(t)^2}{c^2t^2}, \frac{R_0^2}{c^2t^2}\right) \qquad (t_1 < t < T)$$

Here t_1 is a root of Eq. $R_0 + R(t) = ct$.

$$S(t, \tau) = \frac{1}{2\pi R(\tau)\sqrt{R(t)R(\tau)}}\left[K(k_+)\left(1 - \frac{\vartheta_+}{k_+^2}\right) + E(k_+)\frac{\vartheta_+}{k_+^2(1-k_+^2)}\right]$$

$$\{0 < t < t_1,\ 0 < \tau < t\}, \qquad \{t_1 < t < T,\ \lambda(t) < \tau < t\}$$

$$k_+ = \left(1 - \frac{[R(t) + R(\tau)]^2 - c^2(t-\tau)^2}{4R(t)R(\tau)}\right)^{1/2}, \qquad \vartheta_+ = \frac{R(\tau)^2 + c^2(t-\tau)^2 - R(t)^2}{4R(t)R(\tau)}$$

$$S(t, \tau) = \frac{1}{2\pi R(\tau)\sqrt{R(t)R(\tau)}\,k_+}\left[K(1/k_+) + E(1/k_+)\frac{\vartheta_+k_+^2}{1-k_+^2}\right]$$

$$\{t_1 < t < T,\ 0 < \tau < \lambda(t)\}$$

The function $\lambda(t)$ is defined by Eq. $c(t - \lambda) = R(t) + R(\lambda)$.

The absolute term of the integral equation for $B(t)$ assumes the value H_0 for $t = 0$ and has a logarithmic singularity for $t = t_1$. For $t \gg t_1$ we have Formula

$$B(t) \sim -H_0 (R_0^2 / c^2t^2) \{1 + O(R_0^2 / c^2t^2)\}$$

The function $S(t, \tau)$ has a singularity for $\tau = \lambda(t)$ which is a superposition of a logarithmic singularity and a simple pole. The physical meaning of the function $\lambda(t)$ is simple(*). If the perturbation begins to propagate from the cavity boundary at the instant $\lambda(t)$, then the wave reflected from the cylinder axis returns to the wall at the instant t.

Thus, the initial problem in the axisymmetrical case reduces to a Volterra integral Eq. (**) of the second kind (3.7) in the boundary function.

Let $L(t, \tau) = 2cR(\tau)(1 - v^2(\tau))S(t, \tau)$. Eq. (3.7) then becomes

$$H^*(t) = B(t) + \int_0^t L(t, \tau) H^*(\tau)\, d\tau \qquad (3.10)$$

For $t < t_1$ the kernel of Eq. (3.10) is continuous, and the solution $H^*(t)$ can be represented as a Neumann series,

$$H^*(t) = B(t) + \sum_{n=1}^{\infty} \int_0^t L^n(t, \tau) B(\tau)\, d\tau$$

where $L^n(t, \tau)$ are interated kernels from $L(t, \tau)$.

For $t > t_1$ Eq. (3.10) becomes singular, since $L(t, \tau)$ has a singularity in τ at the point $\tau = \lambda(t)$. Here the integral in the right-hand side of the equation represents the principal value.

If we know the boundary function $H^*(t)$, we can find the magnetic field distribution $H(r, t)$. This requires the use of inversion Formula (3.2).

The distribution of $H(r, t)$ can be written as

$$H(r, t) = B_*(r, t) + c \int_0^t H^*(\tau) R(\tau) [1 - v(\tau)^2] S_*(r, t, \tau)\, d\tau$$

$$B_*(r, t) = H_0 \qquad \{0 < t < c^{-1}(R_0 - r),\ 0 < r < R(t)\}$$

$$B_*(r, t) = H_0 \left(1 + \frac{4}{\pi} \sqrt{R_0 r} \int_0^{k^*} \left[K(\varkappa)\left(\frac{\vartheta_*}{\varkappa^2} - 1\right) - E(\varkappa) \frac{\vartheta_*}{\varkappa^2 (1 - \varkappa^2)}\right] \frac{\varkappa}{\nu_*} d\varkappa\right)$$

$$\{0 < r < R(t),\ (R_0 - r)c^{-1} < t < (R_0 + r)c^{-1}\}$$

$$k_* = \left(1 - \frac{(R_0 + r)^2 - c^2t^2}{4R_0 r}\right)^{1/2}, \qquad \nu_* = \sqrt{(R_0 - r^2) + 4\varkappa^2 R_0 r}, \quad \vartheta_* = \frac{\nu_*^2 + R_0^2 - r^2}{4R_0 r}$$

$$B_*(r, t) = -H_0 \frac{R_0^2}{c^2t^2} F_4\left(\frac{3}{2}, 1; 1, 2; \frac{r^2}{c^2t^2}, \frac{R_0^2}{c^2t^2}\right) \quad \{0 < r < R(t),\ (R_0 + r)c^{-1} < t < T\}$$

$$S_*(r, t, \tau) = 0 \qquad \{0 < r < R(\tau) - c(t - \tau),\ 0 < t < R_0 c^{-1},\ 0 < \tau < t\}$$

$$S_*(r, t, \tau) = \frac{1}{2\pi R(\tau) \sqrt{R(\tau) r}} \left[K(k_-)\left(1 - \frac{\vartheta_-}{k_-^2}\right) + E(k_-) \frac{\vartheta_-}{k_-^2 (1 - k_-^2)}\right]$$

$$\{R(\tau) - c(t - \tau) < r < R(t),\ 0 < t < R_0 c^{-1},\ 0 < \tau < t\}$$

$$\{0 < r < R(t),\ R_0 c^{-1} < t < T,\ \omega < \tau < t\}$$

$$k_- = \left(1 - \frac{[r + R(\tau)]^2 - c^2(t - \tau)^2}{4rR(\tau)}\right)^{1/2}, \quad \vartheta_- = \frac{c^2(t - \tau)^2 - r^2 + R(\tau)^2}{4R(\tau) r}$$

$$S_*(r, t, \tau) = \frac{1}{2\pi k_- R(\tau) \sqrt{R(\tau) r}} \left[K(1/k_-) + E(1/k_-) \frac{\vartheta_- k_-^2}{1 - k_-^2}\right]$$

$$\{0 < r < R(t),\ R_0 c^{-1} < t < T,\ 0 < \tau < \omega\}$$

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*) A similar function arises in the plane problem (Formula (2.6)).

**) In the plane case the integral equation degenerates into a functional equation since its kernel has the form of Dirac's delta.

The function $\omega = \omega(t - r/c)$ can be found from Eq. $c(t - \omega) = r + R(\omega)$.

The above formulas imply that the front of the first wave reflected from the axis of the cylinder contains a singularity.

The method of integral representations is a more versatile way of solving problems on the magnetic field compression than is the method of characteristics. Thus, for example, the solutions of problems with axial symmetry or problems taking account of the dissipation of electromagnetic field energy cannot be represented as simple waves, and only the method of integral representations can be used in dealing with them. The latter method can also be applied to a certain class of moving boundary problems in which the boundary of the domain acts as a screen eliminating interaction between the exterior and interior portions of the system

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